(27) 8.1 *Integration By Parts(C)*

Example 1 Evaluate $\int \sin(\ln x)\,dx$ 18 December 1999

Solution

$I = \int \sin(\ln x)\,dx$

Let $t = \ln x \qquad \Rightarrow e^t = x \qquad \Rightarrow\ e^t\,dt = dx$

$I = \int e^t\ \sin t\,dt$

$u = e^t \qquad\qquad dv = \sin t$

$du = \ e^t\,dt \qquad\qquad v = -\cos t$

$I = uv - \int v\,du$

$I = \ -e^t\ \cos t + \int e^t\ \cos t\ \ dt \qquad \rightarrow \boxed{1}$

$I_1 = \int e^t\ \cos t\ \ dt$

$u = e^t \qquad\qquad dv = \cos t$

$du = \ e^t\,dt \qquad\qquad v = \ \sin t\ \ dt$

$I_1 = e^t \sin t - \int e^t \sin t\ \ dt = e^t \sin t - I \qquad \rightarrow \boxed{2}$

$\boxed{2}$ in $\boxed{1}$

$I = \ -e^t\ \cos t + e^t \sin t - I$

$2I = e^t \sin t - e^t\ \cos t + c_1$

$I = \frac{1}{2}\ e^t \sin t - \frac{1}{2} e^t\ \cos t + c\ = \frac{1}{2}\ x \sin(\ln x) - \frac{1}{2}\ x \cos(\ln x) + c$

Example 2 Evaluate $\int \frac{\sin^{-1}(e^x)}{e^{-x}}\,dx$ 43 May 2007

Solution

$I = \int \frac{\sin^{-1}(e^x)}{e^{-x}}\,dx$

Let $t = e^x \qquad \Rightarrow dt = e^x\,dx \qquad \Rightarrow dt = \frac{1}{e^{-x}}\,dx$

$I = \int \sin^{-1} t\ \ dt$

$u = \sin^{-1} t \qquad\qquad dv = dt$

$du = \frac{1}{\sqrt{1-t^2}} \qquad\qquad v = t$

$I = uv - \int v\,du$

$I = t\ \sin^{-1} t - \int \frac{t}{\sqrt{1-t^2}}\ dt = t\ \sin^{-1} t + \frac{1}{2}\int \frac{-2t}{\sqrt{1-t^2}}\ dt = t\ \sin^{-1} t + \frac{1}{2}\cdot 2\sqrt{1-t^2}\ + c$

$\ = \ e^x \sin^{-1} e^x + \sqrt{1-e^{2x}}\ + c$

Example 3

Evaluate $\int 3^x \tan^{-1} 3^x \, dx$ 30 July 2003

Solution

$$I = \int 3^x \tan^{-1} 3^x \, dx$$

Let $t = 3^x \Rightarrow dt = 3^x \ln 3 \; dx \Rightarrow \frac{1}{\ln 3} dt = 3^x \; dx$

$$I = \frac{1}{\ln 3} \int \tan^{-1} t \; dt$$

$u = \tan^{-1} t \qquad dv = \frac{1}{\ln 3} dt$

$du = \frac{1}{1+t^2} \qquad v = \frac{1}{\ln 3} t$

$$I = uv - \int v \, du$$

$$I = \frac{1}{\ln 3} t \; \tan^{-1} t - \frac{1}{\ln 3} \int \frac{t}{1+t^2} \; dt = \frac{1}{\ln 3} t \; \tan^{-1} t - \frac{1}{2\ln 3} \ln(1+t^2) + c$$

$$= \frac{1}{\ln 3} 3^x \; \tan^{-1} 3^x - \frac{1}{2\ln 3} \ln(1+3^x) + c$$

Example 4

Evaluate $\int x \tan^{-1} x \, dx$ 8 May 1997

Solution

$$I = \int x \tan^{-1} x \, dx$$

$u = \tan^{-1} x \qquad dv = x \, dx$

$du = \frac{1}{1+x^2} \qquad v = \frac{1}{2} x^2$

$$I = uv - \int v \, du$$

$$I = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} \; dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{1+x^2-1}{1+x^2} \, dx$$

$$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2} \right) dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int 1 - \frac{1}{1+x^2} \; dx$$

$$= \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + c$$

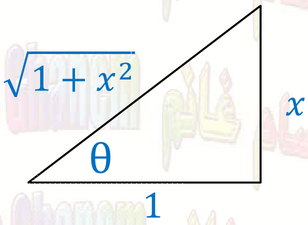
Example 5

Evaluate the following integral $\int \frac{x\tan^{-1}x}{\sqrt{1+x^2}}\,dx$ 33 may 2004 A

Solution

$x = \tan\theta \qquad dx = \sec^2\theta \; d\theta$

$\tan\theta = \frac{x}{1}$



$\theta = \tan^{-1}(x)$

$I = \int \frac{x\tan^{-1}x}{\sqrt{1+x^2}}dx = \int \frac{\theta\tan\theta}{\sqrt{1+\tan^2\theta}}\sec^2\theta\, d\theta = \int \frac{\theta\tan\theta}{\sec\theta}\sec^2\theta\, d\theta = \int \theta\tan\theta\;\sec\theta\;d\theta$

$u = \theta \qquad dv = \tan\theta\sec\theta$

$du = d\theta \qquad v = \sec\theta$

$I = uv - \int v\,du$

$I = \theta\;\sec\theta - \int \sec\theta\;d\theta = \theta\;\sec\theta - \ln|\sec\theta + \tan\theta| + c = \sqrt{1+x^2}\;\tan^{-1}x - \ln\left|\sqrt{1+x^2}+x\right| + c$

Example 6

Evaluate $\int \ln(x^2+2x+2)\,dx$

Solution

$I = \int \ln(x^2+2x+2)\,dx$

$u = \ln(x^2+2x+2) \qquad dv = dx$

$du = \frac{2x+2}{(x^2+2x+2)}\,dx \qquad v = x$

$$\begin{array}{r|l} & \;2 \\ \hline x^2+2x+2 & 2x^2+2x \\ & \underline{\;_-\;2x^2\;_-\;+4x\;_-\;+4} \\ & \quad -2x-4 \end{array}$$

$I = uv - \int v\,du$

$I = x\;\ln(x^2+2x+2) - \int \frac{2x^2+2x}{x^2+2x+2}\;dx$

$\frac{2x^2+2x}{x^2+2x+2} = 2 - \frac{2x+4}{x^2+2x+2} = 2 - \frac{2x+2}{x^2+2x+2} - \frac{2}{x^2+2x+2} = 2 - \frac{2x+2}{x^2+2x+2} - \frac{2}{(x+1)^2+1}$

$I = x\ln(x^2+2x+2) + 2x - \ln(x^2+2x+2) - 2\tan^{-1}(x+1) + c$

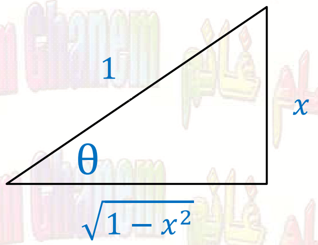
Example 7 Evaluate the integral $\int 2x\cos^{-1}\sqrt{1-x^2}\,dx$ 17 July 1999

Solution

$$I = \int 2x\cos^{-1}\sqrt{1-x^2}\,dx$$

$$x = \sin\theta \qquad dx = \cos\theta \;\; d\theta$$

$$\sin\theta = \frac{x}{1}$$



$$\theta = \sin^{-1} x$$

$$I = \int 2\sin\theta\cos^{-1}\sqrt{1-\sin^2\theta}\cos\theta \;\; d\theta = \int \cos^{-1}\sqrt{\cos^2\theta}\cdot 2\sin\theta\cos\theta \;\; d\theta$$

$$= \int \cos^{-1}(\cos\theta)\sin 2\theta \;\; d\theta \;= \int \theta\sin 2\theta \;\; d\theta$$

$$u = \theta \qquad dv = \sin 2\theta \;\; d\theta$$

$$du = d\theta \qquad v = \frac{-1}{2}\cos 2\theta$$

$$I = uv - \int v\,du$$

$$I = \frac{-1}{2}\theta\;\cos 2\theta + \frac{1}{2}\int \cos 2\theta \;\; d\theta$$

$$= \frac{-1}{2}\theta\;\cos 2\theta + \frac{1}{4}\sin 2\theta \;+ c = \frac{-1}{2}\theta\,(1-2\sin^2\theta) + \frac{1}{2}\sin\theta\cos\theta \;+ c$$

$$= \frac{-1}{2}\sin^{-1}x\;(1-2x^2) + \frac{1}{2}\,x\sqrt{1-x^2} \;+ c$$

Homework

1	Evaluate the integral	$\int \frac{1}{x^2} \cos^{-1} \frac{1}{x} \, dx$	27 December 2002
2	Evaluate the integral	$\int \tan^{-1} \sqrt{x} \, dx$	
3	Evaluate the integral	$\int \sec^{-1} \sqrt{x} \, dx$	
4	Evaluate the integral	$\int 2x \ln(x+1) \, dx$	17 July 1999
5	Evaluate the integral	$\int_0^1 \ln(1+x^2) dx$	
6	Evaluate the integral	$\int x^3 \arctan x \quad dx$ (3 points)	37 August 7, 2010
7	Evaluate the integral	$\int x \cos^2 x \, dx$	
8	Evaluate the integral	$\int \frac{\ln(x+1)}{\sqrt{x+1}} dx$	
9	Evaluate the integral	$\int e^{-x} \ln(1+e^x) \, dx$	
10	Evaluate the integral	$\int 2x \ln(x^3+x) \, dx$	12 December 1997
11	Evaluate the integral	$\int \sin^{-1} \sqrt{x} \, dx$	15 December 1998
12	Evaluate the integral	$\int x^3 \; \tan^{-1} x^2 \;\, dx$	19 May 2000
13	Evaluate the integral	$\int \frac{\ln 2x}{x^2} dx$	23 May 2001
14	Evaluate the integral	$\int 2x \; \sec^{-1} x \, dx$	28 May 2003
15	Evaluate the integral	$\int e^{2x} \ln(1+e^x) \, dx$	39 December 2005
16	Evaluate the integral	$\int x \; \tan^{-1} x^2 \, dx$	
17	Evaluate the integral	$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$	13 May 1998

Homework

18	Evaluate the following. [3.5 pts.] $\int x^2 \tan^{-1} x \; dx$	51 May 13, 2010
19	(5 pts.) Evaluate the following integral $\int x\,[\ln(1+x^2)]^2 \; dx$	38 Jan. 22, 2011
20	Evaluate $\int x^3 \ln \sqrt[3]{x^2+4} \; dx$	37 May 2005
21	Evaluate $\int e^x \sin^2 x \, dx$	
22	Evaluate $\int \frac{x\,e^x}{(x+1)^2} dx$	

20 Evaluate $\int x^3 \ln \sqrt[3]{x^2+4}\ dx$ 37 May 2005

Solution

$I = \int x^3 \ln \sqrt[3]{x^2+4}\ dx = \int \frac{1}{3} x^3 \ln(x^2+4)\ dx$

$u = \ln(x^2+4)$ $dv = \frac{1}{3}\ x^3\ dx$

$du = \dfrac{2x}{x^2+4}$ $v = \frac{1}{12}\ x^4$

$I = uv - \int v\ du$

$I = \frac{1}{12}\ x^4\ \ln(x^2+4) - \frac{1}{6} \int \frac{x^5}{x^2+4}\ dx$

$\frac{x^5}{x^2+4} = x^3 - 4x + \frac{16x}{x^2+4}$

$$\begin{array}{r|l} & x^3-4x \\ \hline x^2+4 & x^5 \\ & \overset{-}{x^5}\ \overset{-}{+4x^3} \\ \hline & -4x^3 \\ & \overset{+}{-4x^3}\ \overset{+}{-16x} \\ \hline & +16x \end{array}$$

$I = \frac{1}{12}\ x^4\ \ln(x^2+4) - \frac{1}{6} \int x^3 - 4x\ + \frac{16x}{x^2+4} = \frac{1}{12}\ x^4\ \ln(x^2+4) - \frac{1}{6} \int x^3 - 4x\ + 8 \cdot \frac{2x}{x^2+4}$

$= \frac{1}{12}\ x^4\ \ln(x^2+4) - \frac{1}{24}\ x^4 - \frac{1}{3}\ x^2 - \frac{4}{3}\ \ln(x^2+4) + c$

21 Evaluate $\int e^x \sin^2 x\, dx$

Solution

$I = \int e^x \sin^2 x\, dx = \frac{1}{2} \int e^x (1 - \cos 2x)\, dx$

$= \frac{1}{2} \int e^x - e^x \cos 2x\, dx = \frac{1}{2}\ e^x - \frac{1}{2} \int e^x \cos 2x\ \ dx$

$I_1 = \int e^x \cos 2x\ \ dx$

$u = \cos 2x$ $dv = e^x\ dx$

$du = -2 \sin 2x\ \ dx$ $v = e^x$

$I_1 = uv - \int v\ du$

$I_1 = e^x \cos 2x + \int 2e^x \sin 2x\ \ dx \qquad \rightarrow \boxed{1}$

$I_2 = \int 2e^x \sin 2x\ \ dx$

$u = \sin 2x$ $dv = 2e^x\ dx$

$du = 2 \cos 2x\ \ dx$ $v = 2e^x$

$I_2 = 2e^x \sin 2x - 4 \int e^x \cos 2x\ \ dx$

$I_2 = 2e^x \sin 2x - 4\, I_1 \qquad \rightarrow \boxed{2}$

$\boxed{2}$ in $\boxed{1}$

$I_1 = e^x \cos 2x + 2e^x \sin 2x - 4\, I_1$

$5\, I_1 = e^x \cos 2x + 2e^x \sin 2x$

$I_1 = \frac{1}{5} e^x \cos 2x + \frac{2}{5} e^x \sin 2x + c_1$

$I = \frac{1}{2}\ e^x - \frac{1}{10} e^x \cos 2x - \frac{1}{5} e^x \sin 2x + c$

22 Evaluate $\int \frac{x\,e^x}{(x+1)^2}\,dx$

Solution

$$\frac{x}{(x+1)^2} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2}$$

$$A(x+1) + B = x$$

at $x = -1 \quad \Rightarrow B = -1$

at $x = 0 \quad \Rightarrow A - 1 = 0 \quad \Rightarrow A = 1$

$$\frac{x}{(x+1)^2} = \frac{1}{(x+1)} - \frac{1}{(x+1)^2}$$

$$\therefore I = \int \frac{x\,e^x}{(x+1)^2}\,dx = \int e^x \cdot \frac{x}{(x+1)^2}\,dx = \int e^x\left(\frac{1}{(x+1)} - \frac{1}{(x+1)^2}\right)dx$$

$$= \int \frac{e^x}{(x+1)}\,dx - \int \frac{e^x}{(x+1)^2}\,dx \qquad \rightarrow \boxed{1}$$

$$I_1 = \int \frac{e^x}{(x+1)}\,dx$$

$u = \frac{1}{(x+1)} \qquad\qquad dv = e^x\,dx$

$du = \frac{-1}{(x+1)^2}\,dx \qquad\qquad v = e^x$

$$I = uv - \int v\,du$$

$$I_1 = \frac{e^x}{(x+1)} + \int \frac{e^x}{(x+1)^2}\,dx \qquad \rightarrow \boxed{2}$$

$\boxed{2}$ in $\boxed{1}$

$$I = \frac{e^x}{(x+1)} + \int \frac{e^x}{(x+1)^2}\,dx - \int \frac{e^x}{(x+1)^2}\,dx = \frac{e^x}{(x+1)} + C$$